SECTION 13.4: THE CROSS PRODUCT

We've already learned one way to multiply two vectors - the dot product. In this section, we learn a second way.

DEFINITION: Given vectors $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$, the **cross product** of $\vec{v}$ and $\vec{w}$ is given by

$$\vec{v} \times \vec{w} = \langle v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1 \rangle$$

There is another (easier?) way to arrive at the cross product using the concept of **determinants**.

DETERMINANTS OF 2×2 **MATRICES:** For a 2×2 matrix,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

EXAMPLE 1: $\begin{vmatrix} 4 & -3 \\ 2 & 1 \end{vmatrix} = (4)(1) - (-3)(2) = 4 + 6 = 10$

DETERMINANTS OF 3×3 **MATRICES:** For a 3×3 matrix:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

EXAMPLE 2:

$$\begin{aligned} \begin{vmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{vmatrix} &= (3) \begin{vmatrix} -1 & 5 \\ 1 & 4 \end{vmatrix} - (1) \begin{vmatrix} 0 & 5 \\ 2 & 4 \end{vmatrix} + (2) \begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix} \\ &= 3((-1)(4) - (5)(1)) - ((0)(4) - (5)(2)) + 2((0)(1) - (-1)(2)) = -13 \end{aligned}$$

ALTERNATIVE DEFINITION: Given vectors $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$:

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

NOTE 1: Since cross product of two vectors is a **vector**, the cross product is also called the **vector** product.

NOTE 2: The cross product is only defined for 3D vectors. That being said, we can always regard a 2D vector as living in 3-space: e.g., if we wanted the cross product of $\vec{v} =< 2, -1 >$ and $\vec{w} =< 1, 3 >$, we can find the cross product of $\vec{v} =< 2, -1, 0 >$ and $\vec{w} =< 1, 3, 0 >$ instead.

EXAMPLE 3: Let $\vec{v} =< 1, 2, -3 >$ and $\vec{w} =< 2, 0, 1 >$.

1. Find and simplify: $\vec{v} \times \vec{w}$

Ans: $\vec{v} \times \vec{w} = \langle 2, -7, -4 \rangle$

2. Find and simplify $\vec{v} \cdot (\vec{v} \times \vec{w})$ and $\vec{w} \cdot (\vec{v} \times \vec{w})$. What does this mean geometrically?

Ans: $\vec{v} \cdot (\vec{v} \times \vec{w}) = \vec{w} \cdot (\vec{v} \times \vec{w}) = 0.$

3. Find and simplify $\vec{w} \times \vec{v}$ and compare your answer to $\vec{v} \times \vec{w}$.

Ans: $\vec{w} \times \vec{v} = \langle -2, 7, 4 \rangle = -(\vec{v} \times \vec{w})$

PROPERTIES OF THE CROSS PRODUCT

- **ANTICOMMUTATIVE:** $\vec{w} \times \vec{v} = -(\vec{v} \times \vec{w})$
- **DISTRIBUTIVE:** $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
- **SCALARS FLOAT:** $\vec{v} \times (k\,\vec{w}) = k\,(\vec{v} \times \vec{w}) = (k\,\vec{v}) \times \vec{w}$
- **ORTHOGONALITY TO FACTORS:** $\vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$.
- **RELATION TO MAGNITUDE:** $\|\vec{v} \times \vec{w}\| = \|\vec{v}\|\,\|\vec{w}\| \sin(\theta)$ where θ is the angle between $\vec{v}$ and $\vec{w}$.
  NOTE: This means $\|\vec{v} \times \vec{w}\|$ is the area of the parallelogram determined by $\vec{v}$ and $\vec{w}$.
- **DETECTS PARALLEL:** If $\vec{v}$ and $\vec{w}$ are nonzero vectors, $\vec{v} \times \vec{w} = \vec{0}$ if and only if $\vec{v}$ and $\vec{w}$ are parallel.

EXAMPLE 4: Find the area of the parallelogram with vertices $(0,0)$, $(1,2)$, $(-2,3)$, $(-1,5)$.

Ans: $7\,\text{units}^2$

CROSS PRODUCTS OF UNIT VECTORS:

- $\hat{\imath} \times \hat{\jmath} =$
- $\hat{\jmath} \times \hat{k} =$
- $\hat{k} \times \hat{\imath} =$
- $\hat{\jmath} \times \hat{\imath} =$
- $\hat{k} \times \hat{\jmath} =$
- $\hat{\imath} \times \hat{k} =$

EXAMPLE 5: Show the cross product is, in general, **not** associative by computing: $\hat{\imath} \times \left(\hat{\imath} \times \hat{\jmath}\right)$ and $\left(\hat{\imath} \times \hat{\imath}\right) \times \hat{\jmath}$.

EXAMPLE 6: For $\vec{v} =< 1, 2, -3 >$ and $\vec{w} =< 2, 0, 1 >$, find $\vec{v} \times \vec{w}$ by writing $\vec{v} = \hat{\imath} + 2\hat{\jmath} - 3\hat{k}$ and $\vec{w} = 2\hat{\imath} + \hat{k}$ and using properties of the cross product and the known values of the cross products of the principal unit vectors.

Ans: $\vec{v} \times \vec{w} = \langle 2, -7, -4 \rangle$

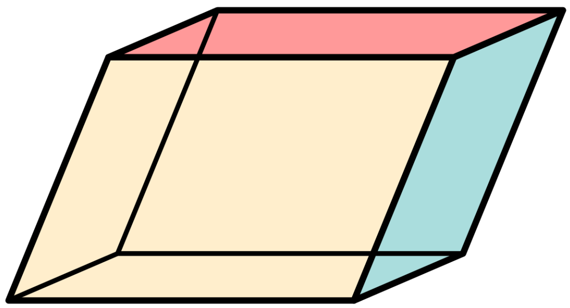
THE TRIPLE SCALAR PRODUCT: Given vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$, and $\vec{w} = \langle w_1, w_2, w_3 \rangle$:

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

is called the **triple scalar product** of $\vec{u}$, $\vec{v}$ and $\vec{w}$.

NOTE: It is true that: $\vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{v} \cdot (\vec{w} \times \vec{u}) = \vec{w} \cdot (\vec{u} \times \vec{v})$

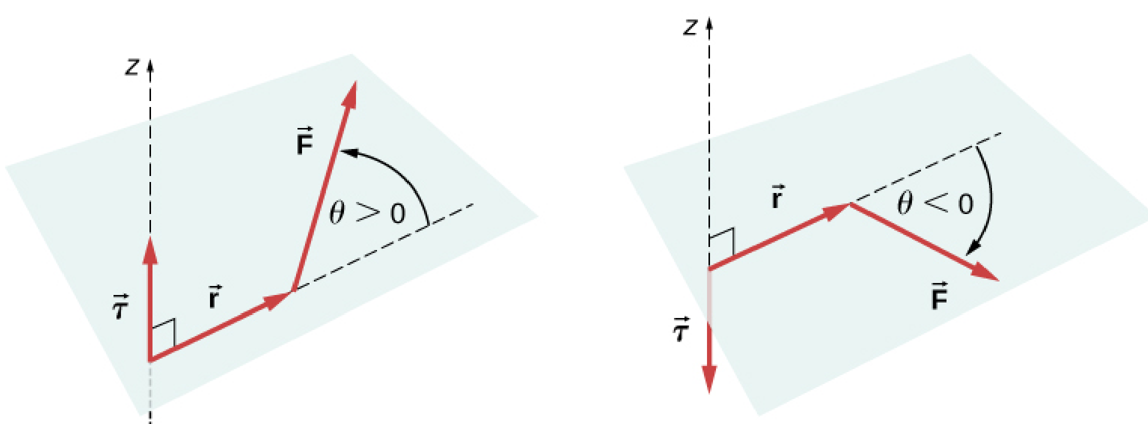
APPLICATION: The volume of a **parallelepiped** determined by vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ is: $|\vec{u} \cdot (\vec{v} \times \vec{w})|$



EXAMPLE 7: Find the volume of the parallelepiped determined by $\vec{u} = \langle -1, 0, 3 \rangle$, $\vec{v} = \langle 2, 1, 0 \rangle$, and $\vec{w} = \langle 1, 3, 2 \rangle$

Ans: 13 units^3

APPLICATION: Torque:



Counterclockwise rotation
the sense of torque:
$+\tau$
(a)

Clockwise rotation
the sense of torque:
$-\tau$
(b)

$$\vec{\tau} = \vec{r} \times \vec{F}, \qquad \|\vec{\tau}\| = \tau = \|\vec{r} \times \vec{F}\| = \|\vec{r}\| \, \|\vec{F}\| \sin(\theta)$$

HOMEWORK: Section 13.4: 13 - 65 every other odd; 73